A GENERALIZATION OF AN INEQUALITY FROM IMO 2005

NIKOLAI NIKOLOV

The present paper was inspired by the third problem from the IMO 2005. A special award was given to Yurie Boreiko from Moldova for his solution to this problem. It was the first such award in the last ten years. Here is the problem.

Problem. *Let x, y and z be positive real numbers such that $xyz \geq 1$. Prove that*

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \geq 0.$$

The main objective of this paper is to prove the following more general inequality:

Proposition 1. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\prod_{i=1}^{n} x_i \geq 1$. Then, for all $\alpha \geq 1$,*

$$\sum_{i=1}^{n} \frac{x_i^\alpha - x_i}{x_1 + \cdots + x_{i-1} + x_i^\alpha + x_{i+1} + \cdots + x_n} \geq 0. \qquad (1)$$

Remark. We get the mentioned IMO problem by choosing $n = 3$ and $\alpha = \frac{5}{2}$, and applying the result to the numbers x^2, y^2 and z^2.

The statement is trivial for $n = 1$; thus, we assume that $n \geq 2$. We will consider two cases, depending on α. In the first case we assume that $\alpha \leq 2 + \frac{1}{n-1}$. We can then prove a stronger inequality in which all denominators are equal to the sum of the numbers. This was the approach used by Yurie Boreiko. (This idea was also suggested by the Armenian deputy leader Nairi Sedrakyan.) The stronger inequality fails for $\alpha > 2 + \frac{1}{n-1}$. In this case, we will estimate the terms of the sum from below by suitable real numbers which sum to zero and have equal denominators. Thus, we use two different ideas for the two cases. The third problem from IMO 2205 is on the boundary of the toe cases in the proof of Proposition 1. It would be interesting to find a unified and concise approach for the proof of Proposition 1.

Proof of Proposition 1:

Case 1. $1 \le \alpha \le 2 + \dfrac{1}{n-1}$.

We have

$$\frac{x_1^\alpha - x_1}{x_1^\alpha + \sum_{i=2}^n x_i} = \frac{x_1 - \frac{1}{x_1^{\alpha-2}}}{x_1 + \frac{\sum_{i=2}^n x_i}{x_1^{\alpha-1}}} \ge \frac{x_1 - \frac{1}{x_1^{\alpha-2}}}{\sum_{i=1}^n x_i} \qquad (2).$$

(To prove the inequality in (2), consider the cases $x_1 \ge 1$ and $x_1 \le 1$ separately). Using (2) and the analogous inequalities for $x_2, \dots, x_n$, we get (1) from the inequality $\displaystyle\sum_{i=1}^n \left(x_i - \frac{1}{x_i^{\alpha-2}}\right) \ge 0$, that is,

$$\sum_{i=1}^n x_i \ge \sum_{i=1}^n x_i^\beta, \qquad (3)$$

where $\beta = 2 - \alpha \in \left[-\dfrac{1}{n-1}, 1\right]$. Now, to prove (3) we will consider two subcases.

Subcase 1.1. $\beta \in [0, 1]$.

Then x^β (for $x \ge 0$) is concave, and Jensen's Inequality implies

$$\left(\frac{\sum_{i=1}^n x_i}{n}\right)^\beta \ge \frac{\sum_{i=1}^n x_i^\beta}{n}. \qquad (4)$$

On the other hand, $\displaystyle\prod_{i=1}^n x_i \ge 1$, and therefore, $\frac{\sum_{i=1}^n x_i}{n} \ge 1$ by the AM-GM Inequality. Since $\beta \le 1$, we get

$$\frac{\sum_{i=1}^n x_i}{n} \ge \left(\frac{\sum_{i=1}^n x_i}{n}\right)^\beta. \qquad (5)$$

Now (3) follows from (4) and (5).

Subcase 1.2. $\beta \in \left[\dfrac{1}{1-n}, 0\right]$.

Then

$$x_1^\beta \le \prod_{i=2}^n x_i^{-\beta} \le \frac{\sum_{i=2}^n x_i^{\beta(1-n)}}{n-1}$$

by the AM-GM inequality. Adding to this inequality the analogous inequalities for $x_2, \dots, x_n$ yields

$$\sum_{i=1}^n x_i^\beta \le \sum_{i=1}^n x_i^{\beta(1-n)}. \qquad (6)$$

But $0 \le \beta(1-n) \le 1$. Thus

$$\sum_{i=1}^{n} x_i^{\beta(1-n)} \le \sum_{i=1}^{n} x_i \tag{7}$$

by Subcase 1.1. Now (3) follows from (6) and (7).

Case 2. $\alpha \ge 2 + \dfrac{1}{n-1}$.

It suffices to show that

$$\frac{x_1^\alpha - x_1}{x_1^\alpha + \sum_{i=2}^n x_i} \ge \frac{n x_1^\gamma - \sum_{i=1}^n x_i^\gamma}{(n-1)\sum_{i=1}^n x_i^\gamma} \tag{8}$$

for some γ, and then to add to (8) the analogous inequalities for $x_2, \ldots, x_n$.

Subtracting 1 from both sides in (8) gives

$$-\frac{\sum_{i=1}^n x_i}{x_1^\alpha + \sum_{i=2}^n x_i} \ge -\frac{n\sum_{i=2}^n x_i^\gamma}{(n-1)\sum_{i=1}^n x_i^\gamma}$$

or, equivalently,

$$n\frac{x_1^\alpha + \sum_{i=2}^n x_i}{\sum_{i=1}^n x_i} \ge \frac{(n-1)\sum_{i=1}^n x_i^\gamma}{\sum_{i=2}^n x_i^\gamma}.$$

Subtracting n from both sides yields

$$\frac{n x_1 (x_1^{\alpha-1} - 1)}{\sum_{i=1}^n x_i} \ge \frac{(n-1)x_1^\gamma}{\sum_{i=2}^n x_i^\gamma} - 1.$$

Since $\prod_{i=1}^{n} x_i \ge 1$, we have $\prod_{i=1}^{n} x_i^{\frac{\alpha-1}{n}} \ge 1$, and the above inequality will follow from the homogeneous inequality

$$\frac{n x_1}{\sum_{i=1}^n x_i}\left(\frac{x_1^{\alpha-1}}{\prod_{i=1}^{n} x_i^{\frac{\alpha-1}{n}}} - 1\right) \ge \frac{(n-1)x_1^\gamma}{\sum_{i=2}^n x_i^\gamma} - 1.$$

We may now assume that $x_1 = 1$. Hence, we need to show that

$$\frac{n}{1+\sum_{i=2}^n x_i}\left(\frac{1}{G} - 1\right) \ge \frac{1}{A} - 1, \tag{9}$$

where

$$A = \frac{\sum_{i=2}^n x_i^\gamma}{n-1} \quad\text{and}\quad G = \prod_{i=2}^{n} x_i^{\frac{\alpha-1}{n}}.$$

We now choose $\gamma = \dfrac{(n-1)(\alpha-1)}{n}$, which implies that $A \ge G$ by the AM-GM Inequality.

For the proof of (9), we will consider two subcases.

Subcase 2.1. $\prod_{i=2}^{n} x_i \geq 1.$

Then (9) follows from the inequalities $0 \geq \frac{1}{G} - 1 \geq \frac{1}{A} - 1$, and

$$\frac{\sum_{i=2}^{n} x_i}{n-1} \geq \sqrt[n-1]{\prod_{i=2}^{n} x_i} \geq 1; \quad \text{that is,} \quad \frac{n}{1+\sum_{i=2}^{n} x_i} \leq 1.$$

Subcase 2.2. $\prod_{i=2}^{n} x_i \leq 1.$

If $A > 1$, then the left side of (9) is nonnegative, while the right side is negative, and (9) is thus true. Otherwise, $\frac{1}{G} - 1 \geq \frac{1}{A} - 1 \geq 0$, and (9) will follow once we show that

$$\frac{n}{1+\sum_{i=2}^{n} x_i} \geq 1, \quad \text{that is,} \quad \frac{\sum_{i=2}^{n} x_i}{n-1} \leq 1.$$

Since $\alpha \geq 2 + \frac{1}{n-1}$ (that is $\gamma \geq 1$), the function x^γ (for $x \geq 0$) is convex. Jensen's Inequality now implies that

$$1 \geq A \geq \left(\frac{\sum_{i=2}^{n} x_i}{n-1}\right)^{\gamma},$$

and (9) is proven.

Remarks.

a) We already mentioned that the approach in Case 1 is not applicable in Case 2, since (3) is incorrect when $\beta \notin \left[\frac{1}{1-n}, 1\right]$. Furthermore, it follows from Subcase 1.1 that, for $\beta > 1$, the opposite inequality holds. The same inequality follows from Subcase 1.1 for $\beta \leq 1-n$ and $\prod_{i=1}^{n} x_i = 1$ (to see this, consider the reciprocals of $x_1, \dots, x_n$). On the other side for $n \geq 3$, $\beta \in \left(1-n, \frac{1}{1-n}\right)$ and $\prod_{i=1}^{n} x_i = 1$ neither (3) nor its opposite holds true. To this aim, we note that if $x_1 = x^{n-1}$, $x_2 = \cdots = x_n = \frac{1}{x}$, then the difference between the left side and the right side of (3) equals

$$x^{n-1} - x^{\beta(n-1)} + (n-1)\left(\frac{1}{x} - \frac{1}{x^\beta}\right).$$

When $\beta \in (1-n, 1)$ and $x \to +\infty$, this difference tends to $+\infty$, while when $\beta < \dfrac{1}{1-n}$ and $x \to 0^+$ it tends to $-\infty$. In particular, when $\beta \in \left(1-n, \dfrac{1}{1-n}\right)$, this difference can have either sign.

b) Let $\gamma = \dfrac{(n-1)(\alpha-1)}{n}$. Then (8) holds for each $\alpha \geq 1$ only when $n = 2$. Indeed, it follows form (9), which is true for $n = 2$ and arbitrary $\alpha \geq 1$ (check!). On the other side when $\prod_{i=1}^{n} x_i = 1$, then (8) is equivalent to (9). When $n \geq 3$, if one chooses

$$x_2 = \left((n - \frac{3}{2}\right)^{\frac{1}{\gamma}},\ x_3 = \cdots = x_n = \left(\frac{1}{2n}\right)^{\frac{1}{\gamma}},$$

then G and A are less than 1 and independent of α. For $\alpha \to 1^+$, one has $x_2 \to +\infty$ and the left side of (9) tends to 0, while the right side is a fixed positive number. Thus, (9) fails for α close to 1, and does (8). For a fixed $n \geq 3$, one might be interested in finding the least $\alpha_n > 1$ that makes (8) true for each $\alpha \geq \alpha_n$ and to prove that it fails for $1 < \alpha < \alpha_n$.

c) Note that, when $\alpha = 2 + \dfrac{1}{n-1}$, we have $\gamma = 1$ and (8) follows from (2) by the AM-GM inequality for the numbers $x_2, \ldots, x_n$.

d) Having in mind Proposition 1, one could expect that for each $\alpha < 1$ the opposite inequality to (1) will be satisfied. For $n = 1$ it is trivial, and the reader may easily check it for $n = 2$. Unfortunately, for $n \geq 3$ this is not true. For example, if $x_1 = x^{n-1} > 1$, $x_2 = \cdots = x_n = \dfrac{1}{x}$ and $\alpha \to -\infty$, the left side of (1) tends to $n - 1 - \dfrac{x^n}{n-1}$, which is positive for $x^n < (n-1)^2$ (> 1 for $n > 2$). It is interesting to find the least $\alpha_n < 1$ such that the opposite inequality to (1) is true for $\alpha_n \leq \alpha \leq 1$ and to prove that it fails for $\alpha < \alpha_n$. Here we can consider the following.

Proposition 2. *Suppose* $n \geq 2$. *If* $\frac{1}{1-n} \leq \alpha \leq 1$ *and the positive real numbers* $x_1, x_2, \ldots, x_n$ *satisfy* $\prod_{i=1}^{n} x_i \geq 1$, *then*

$$\sum_{i=1}^{n} \frac{x_i^\alpha - x_i}{x_1 + \cdots + x_{i-1} + x_i^\alpha + x_{i+1} + \cdots + x_n} \leq 0.$$

Proof: This follows from (3) and

$$\frac{x_i^\alpha - x_i}{x_1 + \cdots + x_{i-1} + x_i^\alpha + x_{i+1} + \cdots + x_n} \leq \frac{x_i^\alpha - x_i}{\sum_{i=1}^n x_i}.$$

Acknowledgements. The results of this paper were obtained at the Summer Research School (2005) of the High School Students' Institute of Mathematics and Informatics. The author would like to thank the Institute for the opportunity to take part in the work of the School. The author is grateful to Professors Oleg Mushkarov, Peter Popivanov and Ivaylo Kortezov for their useful remarks and comments on this paper.

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria

E-mail address: `nik@math.bas.bg`